The Hardy and Caffarelli-Kohn-Nirenberg Inequalities Revisited

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Abstract

In this paper some important inequalities are revisited. First, as motivation, we give another proof of the Hardy's inequality applying convenient vector fields as introduced by Mitidieri, see [6]. Then, we investigate a particular case of the Caffarelli-Kohn-Nirenberg's inequality. Finally, we study the Rellic's inequality.

Contents

1 Introduction

We begin our study by Hardy's inequality, in fact as motivation. Another proof of this inequality is given applying the original ideas of convenient vector fields as introduced by Mitidieri, see [6]. Although, differently from that paper, we use during the proof the generalized Young's inequality, which gives to us a simple way (a la Calculus) to obtain the best constants in some sort of this inequalities.

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Then, we investigate a particular case of the Caffarelli-Korn-Nirenberg's inequality. In fact, we prove this inequality by an interpolation argument based on two convenient parameter points, see Theorem 3.3. That is, first we prove Lemma 3.1, concerning the Caffarelli-Korn-Nirenberg inequality for $b = a + 1$, applying the idea of convenient vector fields, where the technic gives to us the best constant. After, we show a second lemma for $b = a$, where the Sobolev's inequality is used with a convenient function and, further we apply the result proved in the previous Lemma 3.1. To our knowledge this procedure is completely new.

Finally, we investigate the Rellich's inequality, which is a second order type inequality like a generalization of Hardy's one. Our proof is based on the considered particular case of Caffarelli-Korn-Nirenberg's inequality for $b = a + 1$, with $a = 1$. Again, the best constant is recovered in our analysis due to critical point procedure.

1.1 Main purpose

In this paper, we first consider the following sharp version of the Hardy's inequality

$$\int_{\mathbb{R}^n} \frac{|u(x)|^p}{\|x\|^p}\, dx \leq \left(\frac{p}{n-p}\right)^p \int_{\mathbb{R}^n} \|\nabla u(x)\|^p \, dx,$$

where u is a $C_c^\infty(\mathbb{R}^n)$ function and p is a real number, such that $1 < p < n$. Moreover, for $u \in C_c(\mathbb{R}^n \setminus \{0\})$ and $1 < n < p$, the sharp version of Hardy's turns

$$\int_{\mathbb{R}^n} \frac{|u(x)|^p}{\|x\|^p}\, dx \leq \left(\frac{p}{p-n}\right)^p \int_{\mathbb{R}^n} \|\nabla u(x)\|^p \, dx.$$

In one dimensional case, we show the following version of Hardy's inequality

$$\int_0^\infty \left(\frac{\eta(x)}{x}\right)^p dx < \left(\frac{p}{p-1}\right)^p \int_0^\infty u(x)^p \, dx,$$

where $p > 1$, $u \in C_c^\infty(\mathbb{R}+)$ is a nonnegative and non-identically zero function, and

$$\eta(x) := \int_0^x u(t)\, dt.$$

The reader is addressed to Section 1.2 for the functional notation. Following the original idea of Mitidiere [6], we give a new proof of these inequalities applying a nice and simple technique, which further possibility us to recovery direct the best constant.

Also, we analyze a particular case of the well-known inequality due to Caffarelli-Kohn-Nirenberg, which asserts that

$$\left(\int_{\mathbb{R}^n} \frac{|u(x)|^p}{\|x\|^{bp}} dx\right)^{2/p} \leq C(n,b,p) \int_{\mathbb{R}^n} \frac{\|\nabla u(x)\|^2}{\|x\|^{2a}} dx,$$

where $C(n,b,p)$ is a positive constant, $n \geq 3$, $u \in C_c^\infty(\mathbb{R}^n)$ and also

$$-\infty < a < \frac{n-2}{2}, \quad a \leq b \leq a+1, \quad p = \frac{2n}{n-2+2(b-a)}.$$

Following the same ideas applied to prove the Hardy inequality, first we were able to show the Caffarelli-Kohn-Nirenberg's inequality considered, when $b = a+1$. In this particular case, the sign of a has no influence and we recover the sharp constant, that is

$$C(n,b,2) = \frac{4}{(n-2-2a)^2}.$$

On the other hand, when $b = a$ the sign of a has to be considered and we have differences between the constants, see [3]. Indeed, for $a < 0$ the sharp constant is the well known sharp constant of the Sobolev's inequality, that is

$$C(n,b,2^*) = K(n,2)^2,$$

and when $a > 0$ the constant is

$$C(n,b,2^*) = K(n,2)^2 \left(1 + a\sqrt{C(n,b,2)}\right)^{2-2/n}.$$

The general case of Caffarelli-Kohn-Nirenberg'inequality considered is proved by an interpolation argument between these two previous cases, i.e. $b = a+1$ and $b = a$.

At the end of the paper, we prove the Rellich's inequality, which is a first order generalization of Hardy's inequality when $p = 2$, that is

$$\int_{\mathbb{R}^n} \frac{|u(x)|^2}{\|x\|^4}\, dx \leq \left(\frac{4}{n(n-4)}\right)^2 \int_{\mathbb{R}^n} |\Delta u(x)|^2\, dx,$$

where $u \in C_c(\mathbb{R}^n)$, $n > 4$ and $\left(\frac{4}{n(n-4)}\right)^2$ is the sharp constant.

Remark 1.1. *Considering $p = 2$, the inequality due to Hardy present here is a particular case of the Caffarelli-Kohn-Nirenberg's inequality. Indeed, it is enough to take $b = 1$ and $a = 0$ (and consequently $p = 2$).*

An outline of this paper follows. In the rest of this section we fix some definitions and notation. Moreover, we recall some well-known results. The Hardy's inequality is proved on Section 2. In Section 3, first we prove two Lemmas, which are the Caffarelli-Khon-Niremberg's inequality for $b = a+1$ and $b = a$ respectively. Then, applying an interpolation argument we were able to prove the general case. Finally, we show in Section 4 the Rellich's inequality.

1.2 Functional notation and background

At this point we fix the functional notation used in the paper and recall some well known results.

By dx we denote the Lebesgue measure on $\mathbb{R}^n$. Moreover, we denote by $\|.\|$ and $|.|$ respectively the Euclidean norm in $\mathbb{R}^n$ and the absolute value in $\mathbb{R}$.

We recall the generalized Young's inequality: For $\lambda > 0$ and all $V, W \in \mathbb{R}^n$, we have

$$V \cdot W \leq \lambda^{-p} \frac{\|V\|^p}{p} + \lambda^q \frac{\|W\|^q}{q}, \tag{1.1}$$

where $p, q \geq 1$ satisfying

$$\frac{1}{p} + \frac{1}{q} = 1, \quad \big(\text{or } q = p\,(q-1), \text{ or } p = q\,(p-1)\big).$$

For any $U \subset \mathbb{R}^n$ the set $C_c^\infty(U)$ stands for the space of all C^∞ functions on $\mathbb{R}^n$ whose support is compact and contained in U. The Sobolev space $W^{1,p}(\mathbb{R}^n)$ is the set of all functions in $L^p(\mathbb{R}^n)$ with first derivatives also in $L^p(\mathbb{R}^n)$, $(1 \leq p < \infty)$, where the derivatives should be understood in the usual weak sense.

For $1 \leq p < n$, we set

$$p^* := \frac{np}{n-p},$$

called the Sobolev conjugate of p. Thus the Sobolev's inequality asserts that, for all functions $f \in W^{1,p}(\mathbb{R}^n)$

$$\left(\int_{\mathbb{R}^n} |f(x)|^{p^*} dx \right)^{1/p^*} \leq K(n,p)^2 \left(\int_{\mathbb{R}^n} \|\nabla f(x)\|^p \; dx \right)^{1/p}, \tag{1.2}$$

where $K(n,p)$ is the sharp constant, given by

$$K(n,p) = \frac{1}{2^{1/n}\,\pi^{1/2}\,n} \left(\frac{p-1}{n-p} \right)^{1-1/p} \left(\frac{p}{p-1} \right)^{1/n}$$
$$\times \left(\frac{\Gamma(n/2)\,\Gamma(n)}{\Gamma(n/p)\,\Gamma(n(1-1/p))} \right)^{1/n},$$

and $\Gamma(s)$ is the Gamma-function.

2 The Hardy inequality

The proof of the Hardy's inequality follows with a nice strategy, which allow us to apply the Gauss-Green Theorem. Then, the Young's inequality is used to obtain our result.

2.1 The case $(p \neq n, n > 1)$

Theorem 2.1. *Let u be a function in $C_c^\infty(\mathbb{R}^n)$ and $1 < p < n$. Then,*

$$\int_{\mathbb{R}^n} \frac{|u(x)|^p}{\|x\|^p} dx \leq \left(\frac{p}{n-p}\right)^p \int_{\mathbb{R}^n} \|\nabla u(x)\|^p \, dx, \tag{2.3}$$

where $\left(\frac{p}{n-p}\right)^p$ is the sharp constant.

Proof. 1. First, let $V : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n$ be a smooth vector field, defined by

$$V(x) := \frac{x}{(p-n)\,\|x\|^p}. \tag{2.4}$$

For each $i, j = 1, \ldots, n$, this function verifies

$$\frac{\partial V_i(x)}{\partial x_j} = \frac{1}{(p-n)\,\|x\|^p}\,\delta_{ij} - \frac{p}{(p-n)\,\|x\|^{p-2}}\,x_i\,x_k\,\delta_{jk},$$

where the usual summation convention and Kronecker delta notation is used. Consequently, we have

$$\operatorname{div} V(x) = -\frac{1}{\|x\|^p}.$$

2. Now, the integral on the left side of (2.7) can be rewritten in the following way

$$\int_{\mathbb{R}^n} \frac{|u(x)|^p}{\|x\|^p}\, dx = -\int_{\mathbb{R}^n} |u(x)|^p \operatorname{div} V(x)\, dx.$$

Then applying the Gauss-Green Theorem, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{|u(x)|^p}{\|x\|^p}\, dx &= p\int_{\mathbb{R}^n} \Big(|u(x)|^{p-1}\, V(x)\Big)\cdot(\nabla\,|u(x)|)\; dx \\
&\leq p\int_{\mathbb{R}^n} \frac{\lambda^q}{q}\,|u(x)|^{(p-1)q}\|V(x)\|^q\, dx + \int_{\mathbb{R}^n} \lambda^{-p}\|\nabla u(x)\|^p\, dx,
\end{aligned}$$

where we have used the Young's inequality (1.1). Therefore, it follows that

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{|u(x)|^p}{\|x\|^p}\, dx - \frac{p\,\lambda^q}{q}\int_{\mathbb{R}^n} |u(x)|^p\;\|V(x)\|^q\; dx &\leq \lambda^{-p}\int_{\mathbb{R}^n} \|\nabla u(x)\|^p\, dx \\
\left(1 - \frac{\lambda^q p}{q(n-p)^q}\right)\int_{\mathbb{R}^n} \frac{|u(x)|^p}{\|x\|^p} dx &\leq \lambda^{-p}\int_{\mathbb{R}^n} \|\nabla u(x)\|^p\; dx.
\end{aligned} \tag{2.5}$$

From (2.5) and a simple algebraic manipulation, we obtain

$$\int_{\mathbb{R}^n} \frac{|u(x)|^p}{\|x\|^p}\, dx \leq f(\lambda; n, p, q)\int_{\mathbb{R}^n} \|\nabla u(x)\|^p\; dx,$$

where

$$f(\lambda; n, p, q) := \frac{q(n-p)^q}{\lambda^p \big(q(n-p)^q - \lambda^q p\big)}. \tag{2.6}$$

3. Finally, we proceed to obtain the sharp constant. Fixed n, p and thus q, we set the positive constant $\kappa = q(n-p)^q$. Then, we have

$$f(\lambda) = \frac{\kappa}{\lambda^p \, \kappa - \lambda^{p+q} \, p}.$$

So, we can derive f and make it equal zero, to obtain a minimal point candidate, that is

$$\lambda_0 = \Big(\frac{\kappa}{p+q}\Big)^{1/q}.$$

In fact, a straightforward calculation shows that λ_0 is the point of minimum and

$$f(\lambda_0) = \left(\frac{p}{n-p}\right)^p,$$

which is the sharp constant for the Hardy inequality as we already know before.
□

The same proof could be adapted with minor requirements to prove the following

Theorem 2.2. *Let u be a function in $C_c^\infty(\mathbb{R}^n \setminus \{0\})$ and $p > n > 1$. Then,*

$$\int_{\mathbb{R}^n} \frac{|u(x)|^p}{\|x\|^p} dx \leq \left(\frac{p}{p-n}\right)^p \int_{\mathbb{R}^n} \|\nabla u(x)\|^p \, dx, \tag{2.7}$$

where $\left(\frac{p}{p-n}\right)^p$ is the sharp constant.

2.2 The case $(p > n, n = 1)$

Now we are going to prove a sharp version of the Hardy's inequality in one dimension.

Theorem 2.3. *Let u be a nonnegative function in $C_c^\infty(\mathbb{R}_+)$, which is non-identically zero, $p > 1$ and set*

$$\eta(x) := \int_0^x u(t) \, dt, \qquad \Big(\frac{d\eta(x)}{dx} \equiv \eta'(x) = u(x)\Big).$$

Then, we have

$$\int_0^\infty \frac{\eta(x)^p}{x^p} \, dx < \left(\frac{p}{p-1}\right)^p \int_0^\infty u(x)^p \, dx,$$

where $\left(\frac{p}{p-1}\right)^p$ is the sharp constant.

Proof. First, we observe that $\eta(0) = 0$. The proof follows almost the same lines as before. Indeed, we have

$$\begin{aligned}
\int_0^\infty \eta(x)^p \, \frac{1}{x^p} \, dx &= \frac{1}{1-p} \int_0^\infty \eta(x)^p \, (x^{1-p})' \, dx \\
&= \frac{p}{p-1} \int_0^\infty \left(\frac{\eta(x)}{x} \right)^{p-1} \eta'(x) \, dx,
\end{aligned}$$

where we have integrated by parts and used that

$$\begin{aligned}
\lim_{x \to 0} \eta(x)^p \; x^{1-p} = 0, \qquad & (\eta(0) = 0), \\
\lim_{x \to \infty} \eta(x)^p \; x^{1-p} = 0, \qquad & (p > 1).
\end{aligned}$$

Therefore, we obtain

$$\int_0^\infty \left(\frac{\eta(x)}{x} \right)^p dx = \frac{p}{p-1} \int_0^\infty \left(\frac{\eta(x)}{x} \right)^{p-1} u(x) \, dx. \tag{2.8}$$

Now applying the Young's inequality, it follows that

$$\int_0^\infty \left(\frac{\eta(x)}{x} \right)^p dx < \lambda^q \int_0^\infty \left(\frac{\eta(x)}{x} \right)^p dx + \frac{1}{(p-1)\lambda^p} \int_0^\infty u(x)^p \, dx,$$

i.e.,

$$\int_0^\infty \left(\frac{\eta(x)}{x} \right)^p dx < \frac{1}{(p-1)\lambda^p(1-\lambda^q)} \int_0^\infty u(x)^p dx.$$

One remarks that, the equality happens in the Young's inequality before if, and only if

$$\left(\frac{\eta(x)}{x} \right)^{(p-1)q} = u(x)^p.$$

Therefore, it follows that u must be a positive constant, which is a contradiction since u has compact support and is non-identically zero.

Finally, we define for p and thus q fixed,

$$f(\lambda) = \frac{1}{(p-1)\lambda^p(1-\lambda^q)}.$$

Then, we find that the minimum point of f is $\lambda_0 = q^{-q}$, and moreover the minimal value is

$$f(\lambda_0) = \left(\frac{p}{p-1} \right)^p.$$

Consequently, we obtain the sharp Hardy's inequality, that is

$$\int_0^\infty \left(\frac{\eta(x)}{x} \right)^p dx < \left(\frac{p}{p-1} \right)^p \int_0^\infty u(x)^p \, dx.$$

□

3 The Caffarelli-Kohn-Nirenberg inequality

In order to show the inequality due to Caffarelli-Kohn-Nirenberg, we consider first two lemmas, which are particular cases. The former is this inequality for $b = a + 1$, the second one is for $b = a$. The general result follows by an interpolation argument.

Lemma 3.1. *Let u be a function in $C_c^\infty(\mathbb{R}^n)$, $n \geq 3$,*

$$-\infty < a < \frac{n-2}{2} \quad \textit{and} \quad b = a + 1.$$

Then, we have

$$\int_{\mathbb{R}^n} \frac{|u(x)|^2}{\|x\|^{2b}}\, dx \leq C_{a+1} \int_{\mathbb{R}^n} \frac{\|\nabla u(x)\|^2}{\|x\|^{2a}}\, dx, \tag{3.9}$$

where

$$C_{a+1} = \frac{4}{(n-2-2a)^2}$$

is the sharp constant.

Proof. Let $W : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n$ be a smooth vector value function defined as

$$W(x) := \left(\frac{1}{2b-n}\right) \frac{x}{\|x\|^{2b}}.$$

This vector field is well defined since $n \neq 2b$. Indeed, if $b = n/2$, then we must have $a = (n-2)/2$, which contradicts the hypothesis. Moreover, a straightforward calculation shows that

$$\operatorname{div} W(x) = -\frac{1}{\|x\|^{2b}}.$$

Now, the integral of the left side of (3.9) can be written of the following way

$$\int_{\mathbb{R}^n} \frac{|u(x)|^2}{\|x\|^{2b}}\, dx = -\int_{\mathbb{R}^n} |u(x)|^2 \operatorname{div} W(x)\, dx.$$

Then, applying the Gauss-Green Theorem and using the Young's inequality, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{|u(x)|^2}{\|x\|^{2b}}\, dx &= 2\int_{\mathbb{R}^n} \left(\frac{|u(x)|\ W(x)}{\|x\|^{-a}}\right) \cdot \frac{(\nabla |u(x)|)}{\|x\|^{a}} \\ &\leq \alpha^2 \int_{\mathbb{R}^n} \frac{|u(x)|^2 \|W(x)\|^2}{\|x\|^{-2a}}\, dx + \alpha^{-2} \int_{\mathbb{R}^n} \frac{\|\nabla u(x)\|^2}{\|x\|^{2a}}\, dx,\end{aligned} \tag{3.10}$$

where α is a positive real number. Now, we observe that $b = a + 1$ implies

$$\frac{\|W(x)\|^2}{\|x\|^{-2a}} = \frac{\|x\|^2}{(2b-n)^2 \|x\|^{4a+4}} \frac{1}{\|x\|^{-2a}} = \frac{1}{(n-2-2a)^2 \|x\|^{2b}}.$$

Therefore, we have from (3.10)

$$\left(1 - \frac{\alpha^2}{(n-2-2a)^2}\right) \int_{\mathbb{R}^n} \frac{|u(x)|^2}{\|x\|^{2b}} dx \leq \alpha^{-2} \int_{\mathbb{R}^n} \frac{\|\nabla u(x)\|^2}{\|x\|^{2a}} dx$$

and so

$$\int_{\mathbb{R}^n} \frac{|u(x)|^2}{\|x\|^{2b}} dx \leq f(\alpha; n, a) \int_{\mathbb{R}^n} \frac{\|\nabla u(x)\|^2}{\|x\|^{2a}} dx,$$

where

$$f(\alpha; n, a) = \frac{(n-2-2a)^2}{\alpha^2(n-2-2a)^2 - \alpha^4}. \tag{3.11}$$

Analogously, we set $\kappa = (n-2-2a)^2$ and look for the critical points of $f(\alpha)$. Thus we proceed as before and obtain the critical point

$$\alpha_0 = \sqrt{\frac{\kappa}{2}}$$

and also $f''(\alpha_0) > 0$. Moreover, we have

$$f(\sqrt{\kappa/2}) = \frac{4}{(n-2-2a)^2},$$

which is the best constant for this inequality. $\square$

Now we are going to prove a second lemma, which is the Caffarelli-Kohn-Nirenberg's inequality when $b = a$. In this case, we do not follow the same ideas as before, moreover we do not recover the best constant.

Lemma 3.2. *Let u be a function in $C_c^\infty(\mathbb{R}^n)$, $n \geq 3$,*

$$-\infty < 2\,a < n-2, \quad b = a \quad \text{and} \quad p = \frac{2n}{n-2} = 2^*.$$

Then, we have

$$\left(\int_{\mathbb{R}^n} \frac{|u(x)|^p}{\|x\|^{ap}} dx\right)^{\frac{2}{p}} \leq C_{a^\pm} \int_{\mathbb{R}^n} \frac{\|\nabla u(x)\|^2}{\|x\|^{2a}} dx, \tag{3.12}$$

where

$$C_{a^+} = K(n,p)^2 \left(\frac{n-2}{n-2-2a}\right)^2, \quad C_{a^-} = K(n,p)^2 \left(\frac{n-2-4a}{n-2-2a}\right)^2,$$

and a^+ stands for $a \geq 0$ similarly a^- for $a \leq 0$.

Proof. We begin applying the Sobolev inequality (1.2) for $f(x) = u(x)/\|x\|^a$ and without misunderstanding ($p = 2$). Therefore, we obtain

$$\Big(\int_{\mathbb{R}^n} \frac{|u(x)|^p}{\|x\|^{ap}}\, dx\Big)^{\frac{2}{p}} \leq K(n,p)^2 \int_{\mathbb{R}^n} \Big\|\nabla\Big(\frac{u(x)}{\|x\|^a}\Big)\Big\|^2 dx. \tag{3.13}$$

Now, we analyze the right-hand side of the above inequality. First, we observe that

$$\|\nabla\Big(\frac{u(x)}{\|x\|^a}\Big)\|^2 = \frac{\|\nabla u(x)\|^2}{\|x\|^{2a}} + a^2 \frac{|u(x)|^2}{\|x\|^{2(a+1)}} - \frac{2\,a\,u(x)}{\|x\|^{2a+2}} \nabla u(x)\cdot x,$$

and thus

$$\begin{aligned}
\int_{\mathbb{R}^n} \|\nabla\Big(\frac{u(x)}{\|x\|^a}\Big)\|^2\, dx &= \int_{\mathbb{R}^n} \frac{\|\nabla u(x)\|^2}{\|x\|^{2a}}\, dx \\
&\quad + \int_{\mathbb{R}^n} a^2 \frac{|u(x)|^2}{\|x\|^{2(a+1)}}\, dx - \int_{\mathbb{R}^n} \frac{2\,a\,u(x)}{\|x\|^{2a+2}} \nabla u(x)\cdot x\, dx \\
&= I_1 + I_2 + I_3,
\end{aligned} \tag{3.14}$$

with the obvious notations. For I_1 term there is nothing to do, and for I_2 we apply Lemma 3.1, then we have

$$I_2 \leq a^2\, C_{a+1} \int_{\mathbb{R}^n} \frac{\|\nabla u(x)\|^2}{\|x\|^{2a}}\, dx. \tag{3.15}$$

It remains to consider the I_3 term. We divide in two cases: $a < 0$ and $a > 0$. For the former, we have

$$\begin{aligned}
I_3 &\leq -a\lambda^2 \int_{\mathbb{R}^n} \frac{|u(x)|^2}{\|x\|^{2(a+1)}}\, dx \; - \; a\lambda^{-2} \int_{\mathbb{R}^n} \frac{\|\nabla u(x)\|^2}{\|x\|^{2a}}\, dx \\
&\leq -a\left(\lambda^2\, C_{a+1} + \lambda^{-2}\right) \int_{\mathbb{R}^n} \frac{\|\nabla u(x)\|^2}{\|x\|^{2a}}\, dx,
\end{aligned} \tag{3.16}$$

where we have used Young's inequality (1.1) and Lemma 3.1. Therefore, from (3.13)–(3.16), we obtain

$$\Big(\int_{\mathbb{R}^n} \frac{|u(x)|^p}{\|x\|^{ap}}\, dx\Big)^{\frac{2}{p}} \leq f_-(\lambda; n, a) \int_{\mathbb{R}^n} \frac{\|\nabla u(x)\|^2}{\|x\|^{2a}}\, dx,$$

where

$$f_-(\lambda; n, a) = K(n,p)^2 \Big(1 + a^2 C_{a+1} - a\big(\lambda^2 C_{a+1} + \lambda^{-2}\big)\Big).$$

Analogously, we proceed for the second case when $a > 0$, and obtain

$$\Big(\int_{\mathbb{R}^n} \frac{|u(x)|^p}{\|x\|^{ap}}\, dx\Big)^{\frac{2}{p}} \leq f_+(\lambda; n, a) \int_{\mathbb{R}^n} \frac{\|\nabla u(x)\|^2}{\|x\|^{2a}}\, dx,$$

where

$$f_+(\lambda; n, a) = K(n,p)^2\Big(1 + a^2 C_{a+1} + a\big(\lambda^2 C_{a+1} + \lambda^{-2}\big)\Big).$$

Finally, we proceed to minimize $f_\pm$ with respect to λ. Thus we obtain the minimal point

$$\lambda_0 = \Big(\frac{1}{C_{a+1}}\Big)^{1/4},$$

such that $f'_\pm(\lambda_0) = 0$ and $f''_\pm(\lambda_0) > 0$. Moreover, it follows that

$$f_\pm(\lambda_0) = K(n,p)^2\Big(1 \pm a\sqrt{C_{a+1}}\Big)^2.$$

Consequently, we obtain

$$\bigg(\int_{\mathbb{R}^n} \frac{|u(x)|^p}{\|x\|^{ap}} dx\bigg)^{\frac{2}{p}} \leq K(n,p)^2\left(1 \pm a\sqrt{C_{a+1}}\right)^2 \int_{\mathbb{R}^n} \frac{\|\nabla u(x)\|^2}{\|x\|^{2a}} dx.$$

□

We finish this section with the proof of Caffarelli-Kohn-Nirenberg's inequality.

Theorem 3.3. *Let u be a function in $C_c^\infty(\mathbb{R}^n)$ for $n \geq 3$ and assume that*

$$-\infty \leq a < \frac{n-2}{2}, \quad a \leq b \leq a+1 \quad \text{and} \quad p = \frac{2n}{n-2+2(b-a)}.$$

Then, we have for each $\theta \in [0,1]$

$$\bigg(\int_{\mathbb{R}^n} \frac{|u(x)|^p}{\|x\|^{bp}}\, dx\bigg)^{\frac{2}{p}} \leq C_{a^\pm}^{\alpha}\, C_{a+1}^{\beta} \int_{\mathbb{R}^n} \frac{\|\nabla u(x)\|^2}{\|x\|^{2a}}\, dx, \tag{3.17}$$

where $C_{a^\pm}$ and C_{a+1} are respectively the constants for $b = a$ and $b = a+1$, and

$$\alpha = \frac{2n\,\theta}{(n-2)p}, \quad \beta = \frac{2\,(1-\theta)}{p}.$$

Proof. We are going to obtain (3.17) by interpolation between the two previous results obtained in Lemma 3.1 and Lemma 3.2. First, we write for each $0 < \theta < 1$ the exponent p as

$$p = 2(1-\theta) + 2^*\theta. \tag{3.18}$$

We recall that 2^* is the Sobolev conjugate of 2, i.e. $2^* = 2n/(n-2)$. Since we have

$$p = \frac{2n}{n-2+2(b-a)},$$

it follows from (3.18) after a straightforward algebraic calculation that

$$b = a + 1 - \left(\frac{n\theta}{n - 2 + 2\theta} \right) \tag{3.19}$$

and also

$$b\, p = 2\,(1-\theta)(a+1) + 2^*\,\theta\, a. \tag{3.20}$$

Therefore, from (3.18)–(3.20) we could write

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{|u(x)|^p}{\|x\|^{bp}}\, dx &= \int_{\mathbb{R}^n} \frac{|u(x)|^{2(1-\theta)+2^*\theta}}{\|x\|^{2(1-\theta)(a+1)+2^*\theta a}}\, dx \\ &\leq \left(\int_{\mathbb{R}^n} \frac{|u(x)|^2}{\|x\|^{2(a+1)}}\, dx \right)^{1-\theta} \left(\int_{\mathbb{R}^n} \frac{|u(x)|^{2^*}}{\|x\|^{2^* a}}\, dx \right)^{\theta}, \end{aligned}$$

where we have applied Hölder's inequality with

$$\tilde{p} = \frac{1}{1-\theta} \quad \text{and} \quad \tilde{q} = \frac{1}{\theta}.$$

The proof follows from the Lemma 3.1 and Lemma 3.2, that is

$$\left(\int_{\mathbb{R}^n} \frac{|u(x)|^p}{\|x\|^{bp}} dx \right)^{\frac{2}{p}} \leq C_{a^{\pm}}^{\frac{2^*\theta}{p}}\, C_{a+1}^{\frac{2(1-\theta)}{p}} \int_{\mathbb{R}^n} \frac{\|\nabla u(x)\|^2}{\|x\|^{2a}} dx.$$

□

Remark 3.4. *In the previous theorem, instead of $C_{a^\pm}$ and C_{a+1} we could consider respectively $C(n, b, 2^*)$ and $C(n, b, 2)$. Thus for each $\theta \in [0, 1]$ fixed, we obtain the constant of the Caffarelli-Kohn-Nirenberg's inequality, see [3], that is*

$$C(n, b, 2^*)^{\alpha}\, C(n, b, 2)^{\beta}$$

with α and β given by Theorem 3.3

4 The Rellich inequality

In this last section we prove Rellich's inequality, which is a second order generalization of the Hardy's inequality.

Theorem 4.1. *Let u be a function in $C_c^\infty(\mathbb{R}^n)$ and $n > 4$. Then, we have*

$$\int_{\mathbb{R}^n} \frac{|u(x)|^2}{\|x\|^4}\, dx \leq \left(\frac{4}{n(n-4)} \right)^2 \int_{\mathbb{R}^n} |\Delta u(x)|^2\, dx, \tag{4.21}$$

where $\left(\frac{4}{n(n-4)}\right)^2$ is the sharp constant.

Proof. The proof has 3 parts:

1. First, we use the particular case of Caffarelli-Kohn-Nirenberg inequality for $b = a + 1$, with $a = 1$, so

$$\int_{\mathbb{R}^n} \frac{|u(x)|^2}{\|x\|^4} dx \leq M \int_{\mathbb{R}^n} \frac{\|\nabla u\|^2}{\|x\|^2}\, dx, \tag{4.22}$$

where

$$M := \frac{4}{(n-4)^2}.$$

2. Now, we observe the right side of (4.22), and we can note that

$$M \int_{\mathbb{R}^n} \frac{\|\nabla u\|^2}{\|x\|^2} = M \int_{\mathbb{R}^n} \frac{2\,u}{\|x\|^4}\; x \cdot \nabla u\, dx - M \int_{\mathbb{R}^n} \frac{u}{\|x\|^2}\, \Delta u\, dx.$$

Thus using the Young's inequality, we obtain

$$\begin{aligned} M \int_{\mathbb{R}^n} \frac{\|\nabla u\|^2}{\|x\|^2} \leq{} & \frac{M^2}{2\lambda^2} \int_{\mathbb{R}^n} \frac{\|\nabla u\|^2}{\|x\|^2} + \frac{M\lambda^2}{2} \int_{\mathbb{R}^n} |\Delta u|^2 \\ & + \frac{M^2}{\mu^2} \int_{\mathbb{R}^n} \frac{\|\nabla u\|^2}{\|x\|^2} + \mu^2 M \int_{\mathbb{R}^n} \frac{\|\nabla u\|^2}{\|x\|^2}, \end{aligned}$$

where λ, $\mu > 0$. So, we have

$$\int_{\mathbb{R}^n} \frac{|u(x)|^2}{\|x\|^4}\, dx \leq P(\lambda, \mu; M) \int_{\mathbb{R}^n} |\Delta u(x)|^2\; dx, \tag{4.23}$$

where

$$P(\lambda, \mu; M) := M\, \frac{\lambda}{2} \left(1 - \frac{M}{2\lambda^2} - \frac{M}{\mu^2} - \mu^2\right)^{-1}.$$

3. Finally, we can proceed to obtain the critical point (λ_0, μ_0) of P. In fact, replacing this critical point in the determinant of the Hessian of P, i.e. matrix H, we have

$$\det H(\lambda_0, \mu_0) \geq 0.$$

Moreover, we obtain

$$P(\lambda_0, \mu_0) = \frac{16}{n^2(n-4)^2}.$$

Consequently, from (4.23), it follows that

$$\int_{\mathbb{R}^n} \frac{|u(x)|^2}{\|x\|^4} dx \leq \frac{16}{n^2(n-4)^2} \int_{\mathbb{R}^n} |\Delta u(x)|^2\, dx.$$

□

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